2017

(October)

MATHEMATICS

(Elective/Honours)

(GHS–31)

(Algebra—II and Calculus—II)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Show that the set of nth roots of unity is a group under multiplication of complex numbers. 4

(b) Prove that a non-empty subset H of a group G is a subgroup of G if and only if $a, b \in H$ implies $a\,b^{-1} \in H$, where b^{-1} is the inverse of b in G. 4

(c) Verify whether the binary operation '$*$' defined on $\mathbb{Q}$ by $a*b=\frac{ab}{2}$ is (i) commutative and (ii) associative. 1+2=3

(d) Show that a group G is Abelian if and only if $(ab)^2=a^2b^2$ for all $a, b \in G$. 4

2. (a) Prove that every group of prime order is cyclic. Is it Abelian? Justify your answer. 3+1=4

(b) State and prove Lagrange's theorem on the order of a finite group. 1+4=5

(c) Show that the remainder on dividing 7^9 by 15 is 7. State the theorem you have used. 3+1=4

(d) Give an example to show that the union of two subgroups of a group may not be a subgroup. 2

UNIT—II

3. (a) Solve the equation

$$x^4+x^3-16x^2-4x+48=0$$

given that the product of two of its roots is 6. 5

(b) Expand $x^5-6x^3+x^2-1$ in powers of $x+1$. 4

(c) If α, β, γ be the roots of the equation $x^3-ax^2-bx-c=0$, find in terms of the coefficients the values of (i) $\Sigma\alpha^2\beta$ and (ii) $\Sigma\alpha^2\beta^2$. 3+3=6

4. (a) Find all the values of $(1+i)^{1/7}$ by De Moivre's theorem. 5

(b) Solve the equation $x^3-3x+1=0$ by Cardan method. 6

(c) Find the equation whose roots are the roots of $x^5+4x^3-x^2+11=0$ each diminished by 3. 4

UNIT—III

5. (a) Prove that if a sequence converges, then its limit is unique. 4

(b) Show that the sequence $\{x_n\}$, where $x_n=\left(1+\frac{1}{n}\right)^n$ is monotonic increasing. Show also that it is bounded. What can you conclude about the convergence of this sequence? 3+2+1=6

(c) Use Cauchy's general principle of convergence to prove that the sequence $\{x_n\}$ converges, when

$$x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$ 5

6. (a) What is an alternating series? Prove that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges if $\{a_n\}$ is positive monotonic decreasing sequence and $a_n \to 0$ as $n \to \infty$. 2+4=6

(b) Test the convergence of the following series (any *two*) : 3×2=6

(i) $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

(ii) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

(iii) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots\cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots\cdots 2n(2n+1)}$

(c) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$. 3

UNIT—IV

7. (a) State and prove Lagrange's mean value theorem of differential calculus. 1+4=5

(b) Show that $\frac{x}{1+x} < \log(1+x) < x$, for all positive real values of x. 4

(c) Show that $x^{1/x} (x > 0)$ is a maximum at $x = e$ and deduce that $e^{\pi} > \pi^{e}$. 3+1=4

(d) Find the points of inflexion, if any, of the curve $x = (\log y)^3$. 2

8. (a) When is a function $f : D \to \mathbb{R}$ said to be continuous at a point (a, b), where $D \subset \mathbb{R}^2$ and $(a, b) \in D$? Test the continuity of the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

at the origin. 1+4=5

(b) Show that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4 + y^2}$$

does not exist. 2

(c) If

$$u = \frac{x^2y^2}{x+y}$$

apply Euler's theorem to find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ and hence deduce that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 6u$$ 2+4=6

(d) State Schwarz's theorem on mixed partial derivative for a real-valued function of two real variables. 2

UNIT—V

9. (a) Expand $f(x) = \sin x$ in a finite series in powers of x with remainder in Cauchy's form. 4

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $F : [a, b] \to \mathbb{R}$ be a function such that $F'(x) = f(x)$, for all $x \in [a, b]$. Show that $\int_a^b f(x)\, dx = F(b) - F(a)$. 6

(c) Show that the area bounded by the parabolas $x^2 = 4y$ and $y^2 = 4x$ is $\frac{16}{3}$ sq unit. 5

10. (a) Evaluate $\iint_C x^2y^2 dx\,dy$, where

$$C = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$ 5

(b) Find the length of the arc of the parabola $y^2 = 16x$ measured from the vertex to an extremity of its latus rectum. 5

(c) Find the volume and the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about x-axis. 5

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2018

(October)

MATHEMATICS

(Elective/Honours)

(GHS–31)

(Algebra—II and Calculus—II)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Show that the set $G = \{1, 2, 3, \cdots, p-1\}$ is a group of order $p-1$, the composition being ordinary multiplication modulo p, p being a prime integer. 6

(b) If G is a finite group and $a \in G$, then prove that *(i)* if $a^m = e$, then $O(a)$ divides m and *(ii)* $O(a) = O(a^{-1})$, where e is the identity of G, $O(a)$ is the order of element a. 3+2=5

(c) For a given element a in a group G, prove that the set

$$N(a) = \{x \in G \mid xa = ax\}$$

is a subgroup of a group G. 4

2. (a) Prove that every subgroup of a cyclic group is cyclic. 4

(b) Show that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in a group G, where $a, b \in G$. 5

(c) If an element a of a group G satisfies $a^2 = a$, then show that $a = e$. 2

(d) If $G = \langle \mathbb{Z}, + \rangle$ be the group of integer under ordinary addition and if $H = n\mathbb{Z}$, then find all the right cosets of H in G, where n is a fixed positive integer. 4

UNIT—II

3. (a) Find the range of the values of k for which the roots of the equation

$$x^4 + 4x^3 - 8x^2 + k = 0$$

are all real. 5

(b) Find the polynomial $f(x+2)$, when

$$f(x) = 4x^5 + 6x^4 - 3x^3 + 5x - 2$$ 5

(c) Solve the equation

$$4x^4 + 8x^3 + 13x^2 + 2x + 3 = 0$$

given that sum of two of the roots is zero. 5

4. (a) Solve the equation

$$x^3 - 15x - 126 = 0$$

by Cardan's method. 5

(b) Use De Moivre's theorem to solve the equation

$$x^7 - 1 = 0$$ 5

(c) If α, β, γ be the roots of the cubic equation

$$x^3 + px^2 + qx + r = 0$$

find the value of $\sum \alpha^3\beta^3$. 5

UNIT—III

5. (a) Prove that a convergent sequence is bounded. Is the converse true? Justify your answer with an example. 3+1=4

(b) State Cauchy's general principle of convergence of a sequence and apply it to show that the sequence $\{x_n\}$ is divergent, if

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$ 2+3=5

(c) Show that the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$$

converges. Find $\lim_{n \to \infty} a_n$. 3+3=6

6. (a) Examine the convergence of the following series (any *two*) : 3×2=6

(i) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \cdot \frac{1}{n}$

(ii) $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{(n+1)x^n}{n^3} + \cdots$

(iii) $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - \sqrt{n^3} \right)$

(b) What is an absolute convergent series? Test the absolute convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(n+1)!}$$ 2+3=5

(c) Determine the region of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 3^n}$$ 4

UNIT—IV

7. (a) State Rolle's theorem and give its geometrical interpretation. 2+2=4

(b) Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin. 3

(c) Verify Lagrange's mean value theorem for the function

$$f(x) = x(x-1)(x-2)$$

in $\left[0, \frac{1}{2}\right]$. 4

(d) Find the asymptotes of

$$xy^2 - y^2 - x^3 = 0$$ 4

8. (a) Show that for the function $f(x, y)$ defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \; x^2 + y^2 \neq 0$$

$\lim_{x \to 0} \lim_{y \to 0} f(x, y)$ and $\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ both exist but are unequal. Also show that

$$\lim_{(x, y) \to (0, 0)} f(x, y)$$

does not exist. 1+1+2=4

(b) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then show that—

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$;

(ii) $x^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} =$

$(1 - 4\sin^2 u)\sin 2u$. 2+4=6

(c) If $u = r^3$, $x^2 + y^2 + z^2 = r^2$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 12r$$ 5

UNIT—V

9. (a) Expand $(1 + x)^m$ in a finite series in power of x with Lagrange's form of remainder. 4

(b) Show that the equation $x^3 + 2x - 8 = 0$ has a root between 1 and 2. Taking 2 as an approximate root and using Newton's method for approximation, find that the root corrects to 3 decimal places. 5

(c) State and prove the fundamental theorem of integral calculus. 1+5=6

10. (a) Find the area enclosed by the curves $x^2 + y^2 = 2ax$ and $y^2 = ax$. 5

(b) Evaluate $\iint xy(x^2 + y^2)\,dxdy$ over the region $R = [0, a; 0, b]$. 5

(c) Find the value of $\int_C (x^2 + y^2)\,dy$, where C is the arc of the parabola $y^2 = 4ax$ between $(0, 0)$ and $(a, 2a)$. 5

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3/EH–29 (iii) (Syllabus–2015)

2016

(October)

MATHEMATICS

(Elective/Honours)

(Algebra—II and Calculus—II)

(GHS–31)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. (a) Prove that the set $\{1, -1, i, -i\}$ is a finite Abelian group of order 4 with respect to multiplication. $[i^2 = -1]$ 3

   (b) Show that every subgroup of a cyclic group is cyclic. 4

(c) Answer the following with justification : 2×3=6

(i) Can an Abelian group have a non-Abelian subgroup?

(ii) Can a non-Abelian group have an Abelian subgroup?

(iii) Can a non-Abelian group have a non-Abelian subgroup?

(d) Prove that intersection of any two subgroups of a group is a subgroup. 2

2. (a) If G is a finite group, show that for each $a \in G$, there exists a positive integer n such that $a^n = e$, where e is the identity element of a group G. 5

(b) Show that any two left cosets of a subgroup H in a group G have the same (finite or infinite) number of elements. 5

(c) Show that an infinite cyclic group has exactly two generators. 5

UNIT—II

3. (a) Solve $x^4 - x^3 + 3x^2 + 31x + 26 = 0$, if one of the roots of the given equation is $2 - 3i$. 5

(b) Find the polynomial $f(x+2)$, when

$$f(x) = x^4 - 3x^3 + 4x^2 - 2x + 1$$ 4

(c) Remove the second term of the equation

$$x^3 + 6x^2 + 12x - 19 = 0$$

and then solve the given equation. 6

4. (a) (i) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, form the equation whose roots are

$$\beta\gamma + \frac{1}{\alpha}, \quad \gamma\alpha + \frac{1}{\beta}, \quad \alpha\beta + \frac{1}{\gamma}$$ 4

(ii) If $z = \cos 2\theta + i \sin 2\theta$
$w = \cos 2\phi + i \sin 2\phi$

show that

$$z^m w^n + \frac{1}{z^m w^n} = 2\cos 2(m\theta + n\phi)$$ 2

(b) If the equation

$$3x^4 + 4x^3 - 60x^2 + 96x - k = 0$$

has four real and unequal roots, show that k must lie between 32 and 43. 4

(c) Solve the equation $x^3 - 18x - 35 = 0$ by Cardan's method. 5

UNIT—III

5. (a) Prove that the convergent sequence is bounded. Is the converse true? Justify with an example. 3+1=4

(b) Show that if $x_n = \frac{3n+1}{n+2}$, then the sequence $\{x_n\}$ is strictly increasing. Is the sequence convergent? Justify your answer. Also find its limit. 3+2+1=6

(c) Define Cauchy sequence. Is the sequence $\{n^2\}$ a Cauchy sequence? Justify your answer. 2+3=5

6. (a) Test the convergence of the following series (any *two*) : 3×2=6

(i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(ii) $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

(iii) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} \right)$

(b) State Leibnitz's theorem for alternating series. Show that $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ converges. 2+3=5

(c) Define a power series. Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. 1+3=4

UNIT—IV

7. (a) State and prove Cauchy's mean value theorem. 1+4=5

(b) Find the asymptotes of the curve

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$$ 5

(c) (i) Find the approximate value of log 10·1 by the use of differentials. Given that $\log_{10} e = 0 \cdot 4343$. 2

(ii) Show that of all rectangles of a given area, the square has the smallest perimeter. 3

8. (a) Let the function be defined by

$$f(x, y) = \begin{cases} \frac{2xy}{\sqrt{x^2 + y^2}}, & \text{when } x^2 + y^2 \neq 0 \\ 0, & \text{when } x = 0 = y \end{cases}$$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. 6

(b) State and prove Euler's theorem on homogeneous function of three variables x, y, z. Applying Euler's theorem to the function $V = \tan^{-1} \frac{x^3 + y^3}{x - y}$, show that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \sin 2V.$$ 1+4+4=9

UNIT—V

9. *(a)* Expand $\log(1+x)$ in a finite series in powers of x with Cauchy's form of remainder. 4

(b) State and prove Taylor's theorem in infinite form with Lagrange's form of remainder. 6

(c) State and prove the fundamental theorem of integral calculus. 1+4=5

10. *(a)* Find the volume of the solid of revolution obtained by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the minor axis. 4

(b) Find the length of arc of the given curves $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$. 5

(c) Evaluate $\int_2^4 \int_{4/x}^{\frac{20-4x}{8-x}} (4-y)\,dy\,dx$ by changing the order of integration. 6

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3/EH–29 (iii) (Syllabus–2015)

2019

(October)

MATHEMATICS

(Elective/Honours)

(GHS–31)

(Algebra—II & Calculus—II)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. *(a)* Show that the set of all positive rational numbers forms an Abelian group under the composition defined by

$$a * b = \frac{ab}{2}$$ 5

(b) If the inverse of a is a^{-1}, then show that the inverse of $a^{-1} = a$. 2

(c) Show that the set consisting of the fourth roots of unity namely $1, -1, i, -i$ form a group with respect to multiplication. 5

(d) Show that if every element of a group G is its own inverse, then G is Abelian. 3

2. (a) Show that if a, b are any two elements of a group G, then $(ab)^2 = a^2b^2$ if and only if G is Abelian. 3

(b) Give an example to show that the union of two subgroups is not necessarily a subgroup. 2

(c) Prove that any two right (left) cosets of a subgroup are either disjoint or identical. 5

(d) Prove that every group of prime order is cyclic. 5

UNIT—II

3. (a) Prove that

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cos\frac{n\pi}{4}$$

n being any positive integer. 5

(b) Solve the following equation by Cardan's method :

$$x^3 + 6x + 7 = 0$$ 5

(c) Solve the equation

$$x^4 - 3x^3 - 5x^2 + 9x - 2 = 0$$

if one of the roots of the given equation is $2+\sqrt{3}$. 5

4. (a) If the sum of the two roots of $x^4 + px^3 + qx^2 + rx + s = 0$ is equal to the sum of its other two roots, then prove that $p^3 + 8r = 4pq$. 5

(b) If

$$x^4 - 14x^2 + 24x - k = 0$$

has four real, unequal roots, prove that k must lie between 8 and 11. 5

(c) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are

$$\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}, \frac{\alpha+\beta}{\gamma^2}$$ 5

UNIT—III

5. (a) Prove that a sequence which is monotonic increasing and bounded above converges to its exact upper bound. 5

(b) Show that the sequence

$$\left\{1-\frac{1}{n}\right\}$$

is bounded above. Is it monotonic? Find its limit, if the limit exists. 5

(c) Prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence. 5

6. (a) Test the convergence of the following series (any *two*) : 3×2=6

(i) $\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\ldots$

(ii) $\frac{1}{1\cdot 2^2}+\frac{1}{2\cdot 3^2}+\ldots$

(iii) Σu_n, where $u_n=\frac{\sqrt{n}}{n^2-1}$

(b) By using Leibnitz's test, prove that the series

$$1-\frac{1}{2}+\frac{1}{3}-\ldots$$

is convergent. 3

(c) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{n^2}{\sqrt{n}}$$ 6

UNIT—IV

7. (a) State and prove Cauchy's mean value theorem. 1+4=5

(b) Verify Rolle's theorem for the function $f(x)=x^2-5x+6$ in $1\le x\le 4$. 3

(c) Find the maximum and minimum value of

$$1+2\sin x+3\cos^2 x,\quad 0\le x\le\frac{1}{2}\pi$$ 3

(d) Show that the rectangle inscribed in a circle has maximum area, when it is a square. 4

8. (a) If $u=\log r$, $r^2=x^2+y^2+z^2$, prove that

$$r^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$$ 6

(b) If $v = f(u)$, u being a homogeneous function of degree n in x and y, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = nu\frac{\partial v}{\partial u}$$ 4

(c) Show that
$$x^3 - 6x^2 + 12x - 3$$
is neither a maximum nor a minimum, when $x = 2$. 5

UNIT—V

9. (a) Expand $\cos x$ in powers of x with Maclaurin's form of remainder. 4

(b) State and prove the fundamental theorem of integral calculus. 6

(c) Find the length of the arc of the parabola $y^2 = 4ax$ measured from the vertex to one extremity of the latus rectum. 5

10. (a) Show that the volume of the solid formed by the rotation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about major axis is $\frac{4}{3}\pi ab^2$ and about minor axis is $\frac{4}{3}\pi a^2 b$. 4

(b) Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point $x = 2$. 5

(c) Evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} x\,dx\,dy$$ 6

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